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Nonlocal Coupled Damage-Plasticity Model Incorporating Functional Forms of Hardening State Variables

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The thermodynamically consistent formulation and the subsequent numerical implementation of a gradient-enhanced, continuum-coupled damage-plasticity model as a constitutive framework to model ill-posed localization problems is presented. The formulation of the elastoplastic-damage behavior of materials is introduced here within a framework that uses functional forms of hardening internal state variables in both damage and plasticity. Various exponential and power law functional forms are studied in this formulation. Gradients of hardening terms are found directly by operating on the respective hardening terms, and numerical methods are used to compute these gradients. The gradient-enhanced measure used in this work is justified by an approximation to nonlocal theory; however, through the expansion of various gradient terms in this nonlinear hardening plasticity model, gradients of both odd and even orders are introduced into the constitutive model. A multifield method is used such that the displacement field is interpolated using standard continuous elements, and higher-order elements (cubic Hermitian) are used for the plastic multiplier and for the damage multiplier to enforce continuity of the second-order gradients. The effectiveness of the model is evaluated by studying the mesh-dependence issue in localization problems through numerical examples.

I. Introduction

A STATE of localized deformation is described by a state in which, after a point of instability, all further deformation concentrates in a small, but finite, region. Because all of the strains localize in this region, the structure outside of this localized zone tends to unload elastically. The width, point of emergence, and angle of inclination of the shear bands can depend on material factors such as the grain diameter (either soil or metal crystal), the distribution and density of fibers in a metal-matrix composite, and the angle of internal friction, and they can depend on structural factors such as the geometry of the body and the loading boundary conditions. Softening behavior typically occurs when the shape of the body and the boundary conditions induce an inhomogeneous state of deformation. The nonuniform shape of the body triggers localization to occur; however, material heterogeneity due to microstructures such as microcracks, microvoids, or dislocations also gives rise to an inhomogeneous state of deformation.

II. Gradient-Enhanced Continuum Mechanics and Thermodynamics

The gradient-enhanced constitutive model is derived using consistent thermodynamics in the same fashion as a classical rate-independent continuum J2 plasticity model [1–3], coupled with a continuum damage model [4–6]. The thermodynamics of irreversible processes followed here will introduce a nonlocal state consisting of state variables [1,7,8] and the corresponding gradients of the state variables.

A. State Variables

The nonlocal coupled plasticity-damage model is developed in which the thermodynamic state at a given point in space and time is completely determined by a given set of state variables. Because this set of state variables includes gradients, the state at a given point is dependent on the behavior surrounding the point (i.e., the state is nonlocal). The set of state variables are separated into a set of observable state variables and a set of internal state variables. The observable state variables used here are the temperature denoted by the scalar T, the total strain denoted by the second-order tensor $\boldsymbol{\varepsilon}$, and the damage tensor denoted by the second-order tensor $\boldsymbol{\varphi}$. For this coupled damage-plasticity model, the hardening internal state variables are unitless, strainlike quantities and are accumulated into a set of plasticity-related measures $\mathbf{V}^p$ and a set of damage-related measures $\mathbf{V}^d$, as follows:

$$\mathbf{V}^p = [r, \boldsymbol{\alpha}]; \qquad \mathbf{V}^d = [\kappa] \tag{1}$$

where the scalar r and the second-order tensor $\boldsymbol{\alpha}$ represent the fluxes of the plasticity isotropic and kinematic hardening, respectively, and the scalar κ represents the flux of the damage isotropic hardening. Sets of conjugate forces, $\mathbf{A}^p$ and $\mathbf{A}^d$, corresponding to these internal state variables are defined such that:

$$\mathbf{A}^p = [R, \mathbf{X}]; \qquad \mathbf{A}^d = [K] \tag{2}$$

where the scalar R [9] associated with r measures the expansion or contraction of the yield surface in the stress space, the second-order tensor $\mathbf{X}$ [10] associated with $\boldsymbol{\alpha}$ measures the movement and distortion of the yield surface, and the scalar K associated with κ measures the expansion or contraction of the damage surface in the stress space. Whereas the internal state variables are unitless, strainlike quantities, the thermodynamic conjugate forces are a set of stresslike quantities that are related to the state variables, because the stress is related to the strain.

As discussed previously, the form of the nonlocal measures used in this work is justified from the discussion of approximating the nonlocal integral equation. To this end, the gradient-enhanced measures $\bar{R}$, $\bar{\mathbf{X}}$, and $\bar{K}$ are used to characterize nonlocal conjugate forces and are given as follows [11]:

$$\bar{R} = R + c_R \nabla^2 R; \qquad \bar{\mathbf{X}} = \mathbf{X} + c_X \nabla^2 \mathbf{X}; \qquad \bar{K} = K + c_K \nabla^2 K \tag{3}$$

The coefficients c_R, c_X, and c_K introduce material length scales and are defined to be a constant proportional to an internal characteristic length squared. For kinematic hardening, the constant

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c_X weights each component of the gradient tensor identically. Because stresses and strains are macrovariables that are computed using the internal state variables of the material, gradient effects are not introduced directly [12] through the strains and stresses by introducing gradient-dependent, nonlocal measures of the stress and strain [13]. Similarly, because the damage measure is a macrovariable similar to the strain, nonlocal measures of $\boldsymbol{\varphi}$ are not introduced in this work.

B. Stress Transformations

For this gradient-enhanced state, additional stress transformation equations are required. Rather than transform both the plasticity conjugate forces and their Laplacians, the nonlocal measure of these hardening conjugate forces defined by Eqs. (3) are transformed. The transformation to the effective configuration of the scalar conjugate force $\bar{R}$ and of the tensorial conjugate force $\bar{\mathbf{X}}$ is performed as follows:

$$\tilde{\bar{R}} = \frac{\bar{R}}{1 - \|\boldsymbol{\varphi}\|}; \qquad \tilde{\bar{\mathbf{X}}} = \mathbf{M}:\bar{\mathbf{X}} \tag{4}$$

where $\tilde{\bar{R}}$ and $\tilde{\bar{\mathbf{X}}}$ are the nonlocal measures of the isotropic and kinematic hardening conjugate forces in the effective configuration. The relative stress tensors can now be defined as follows:

$$\tilde{\boldsymbol{\xi}} = \tilde{\mathbf{s}} - \tilde{\mathbf{X}}; \qquad \boldsymbol{\xi} = \mathbf{s} - \mathbf{X} \tag{5}$$

Because $\bar{\mathbf{X}}$ is a deviatoric tensor, $\mathbf{M}$ can be replaced with $\mathbf{N}$ in Eq. (4). Thus, substituting Eqs. (4) into Eq. (5), the relative stress tensor is written in terms of the effective configuration, as follows:

$$\tilde{\boldsymbol{\xi}} = \mathbf{N}:(\boldsymbol{\sigma} - \mathbf{X}) \tag{6}$$

C. Equations of State

To determine state laws that relate the internal state variable fluxes to their conjugate thermodynamic forces, a thermodynamic potential defined as the Helmholtz free energy is introduced, which is a state function of a thermodynamic system [1,7,8]. This thermodynamic potential is used to describe the current state of energy in the material and is a function of the observable state variables and the internal state variables under consideration:

$$\psi = \psi(\boldsymbol{\varepsilon}^e, T, \boldsymbol{\varphi}, \mathbf{V}^p, \nabla\mathbf{V}^p, \nabla^2\mathbf{V}^p, \mathbf{V}^d, \nabla\mathbf{V}^d, \nabla^2\mathbf{V}^d) \tag{7}$$

where the gradient sets are sets of gradients of the corresponding internal state variables in Eqs. (1). The second law of thermodynamics imposes restrictions on the constitutive relations through the Clausius–Duhem inequality:

$$\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \rho(\dot{\psi} + s\dot{T}) - \mathbf{q}\cdot\frac{\nabla T}{T} \geq 0 \tag{8}$$

where $\boldsymbol{\sigma}$ is the second-order Cauchy stress tensor, ρ is the mass density, $\mathbf{q}$ is the heat flux vector, s is the entropy per unit mass representing the amount of disorder or randomness in a system, ∇T is the temperature gradient, and $\dot{\psi}$ is the time derivative of ψ, which is given as follows:

$$\dot{\psi} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e}:\dot{\boldsymbol{\varepsilon}}^e + \frac{\partial\psi}{\partial T}\dot{T} + \frac{\partial\psi}{\partial\boldsymbol{\varphi}}:\dot{\boldsymbol{\varphi}} + \frac{\partial\psi}{\partial\mathbf{V}^p}\cdot\dot{\mathbf{V}}^p + \frac{\partial\psi}{\partial\nabla\mathbf{V}^p}\cdot\nabla\dot{\mathbf{V}}^p + \frac{\partial\psi}{\partial\nabla^2\mathbf{V}^p}\cdot\nabla^2\dot{\mathbf{V}}^p + \frac{\partial\psi}{\partial\mathbf{V}^d}\cdot\dot{\mathbf{V}}^d + \frac{\partial\psi}{\partial\nabla\mathbf{V}^d}\cdot\nabla\dot{\mathbf{V}}^d + \frac{\partial\psi}{\partial\nabla^2\mathbf{V}^d}\cdot\nabla^2\dot{\mathbf{V}}^d \tag{9}$$

Using this relationship along with the strain decomposition, Eqs. (8) and (17) can be expanded and, assuming processes that independently satisfy this inequality, the following thermodynamic laws are obtained:

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e}; \qquad s = -\frac{\partial\psi}{\partial T}; \qquad \mathbf{Y} = -\frac{\partial\psi}{\partial\boldsymbol{\varphi}} \tag{10}$$

$$\mathbf{A}^p = \rho\frac{\partial\psi}{\partial\mathbf{V}^p}; \qquad \mathbf{A}^d = \rho\frac{\partial\psi}{\partial\mathbf{V}^d} \tag{11}$$

From these state laws, the stress $\boldsymbol{\sigma}$ and the enthalpy s are defined as the conjugate forces corresponding to the state variables $\boldsymbol{\varepsilon}^e$ and T, respectively. Similarly, $\mathbf{Y}$ is defined as the conjugate force corresponding to $\boldsymbol{\varphi}$, and the sets of conjugate forces $\mathbf{A}^p$ and $\mathbf{A}^d$ are defined, which correspond to the sets of internal state variables.

D. Conjugate Forces

Because the internal state variables are selected independently of one another, one can express the analytical form of the Helmholtz free energy, in terms of its internal state variables, as

$$\rho\psi = \tfrac{1}{2}\boldsymbol{\varepsilon}^e:\mathbf{C}^e:\boldsymbol{\varepsilon}^e + W^p(\mathbf{V}^p, \nabla\mathbf{V}^p, \nabla^2\mathbf{V}^p) + W^d(\mathbf{V}^d, \nabla\mathbf{V}^d, \nabla^2\mathbf{V}^d) - \rho T s \tag{12}$$

where the scalar ρ is the material density, and the superscripted e, p, and d imply terms associated with elasticity, plasticity, and damage, respectively. For this form of the Helmholtz free energy and from the state laws, Eqs. (10), the stress is defined in the form of a Hookean relationship, and a relation for the damage measure conjugate force is defined:

$$\boldsymbol{\sigma} = \mathbf{C}^e:\boldsymbol{\varepsilon}^e; \qquad \mathbf{Y} = \frac{1}{2}\boldsymbol{\varepsilon}^{eT}:\frac{\partial\mathbf{C}^e}{\partial\boldsymbol{\varphi}}:\boldsymbol{\varepsilon}^e \tag{13}$$

The terms $W^p(\mathbf{V}^p, \nabla\mathbf{V}^p, \nabla^2\mathbf{V}^p)$ and $W^d(\mathbf{V}^d, \nabla\mathbf{V}^d, \nabla^2\mathbf{V}^d)$ account for energy introduced into the system by hardening terms and their corresponding gradient terms for plasticity and damage, respectively. In general, the energy terms may be introduced as fully coupled for the various hardening terms. However, in this work it is assumed that the energy introduced by the hardening terms is uncoupled:

$$W^p(\mathbf{V}^p, \nabla\mathbf{V}^p, \nabla^2\mathbf{V}^p) = W^r(r) + W^{\nabla r}(\nabla r) + W^{\nabla^2 r}(\nabla^2 r) + W^{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) + W^{\nabla\boldsymbol{\alpha}}(\nabla\boldsymbol{\alpha}) + W^{\nabla^2\boldsymbol{\alpha}}(\nabla^2\boldsymbol{\alpha}) \tag{14}$$

$$W^d(\mathbf{V}^d, \nabla\mathbf{V}^d, \nabla^2\mathbf{V}^d) = W^\kappa(\kappa) + W^{\nabla\kappa}(\nabla\kappa) + W^{\nabla^2\kappa}(\nabla^2\kappa) \tag{15}$$

The energy terms can be written in terms of a power or exponential relationship for the isotropic hardening energy term [1], for the kinematic hardening energy term [14], and for the damage isotropic hardening energy term [14]. Thus, the energy terms can be (but are not required to be) selected to be in the form of *power laws,* as follows:

$$W^r(r) = \frac{H_r}{m_r + 1}r^{m_r+1}; \qquad W^\alpha(\boldsymbol{\alpha}) = \frac{H_\alpha}{m_\alpha + 1}\|\boldsymbol{\alpha}\|^{m_\alpha+1} \qquad W^\kappa(\kappa) = \frac{H_\kappa}{m_\kappa + 1}\kappa^{m_\kappa+1} \tag{16}$$

Or they can be (but, again, are not required to be) selected to be in the form of *exponential laws*, as follows:

$$W^r(r) = R_\infty\left(r + \frac{1}{\gamma_r}e^{-\gamma_r r} - \frac{1}{\gamma_r}\right) \qquad W^\alpha(\boldsymbol{\alpha}) = X_\infty\left(\|\boldsymbol{\alpha}\| + \frac{1}{\gamma_\alpha}e^{-\gamma_\alpha\|\boldsymbol{\alpha}\|} - \frac{1}{\gamma_\alpha}\right) \qquad W^\kappa(\kappa) = K_\infty\left(\kappa + \frac{1}{\gamma_\kappa}e^{-\gamma_\kappa\kappa} - \frac{1}{\gamma_\kappa}\right) \tag{17}$$

In these relationships, H_i, m_i, R_∞, and X_∞ are material and geometrical dependent parameters, where $i = r, \alpha, \kappa$. From the Helmholtz free energy and the state laws, Eqs. (11), the *power law* form of the energy terms result in definitions for the hardening thermodynamic conjugate forces, as follows:

$$R = H_r r^{m_r}; \qquad \mathbf{X} = H_\alpha \|\boldsymbol{\alpha}\|^{m_\alpha - 1}\boldsymbol{\alpha}; \qquad K = H_\kappa r^{m_\kappa} \tag{18}$$

Similarly, the *exponential law* form of the energy terms result in the following relations:

$$R = R_\infty(1 - e^{-\gamma_r r}); \qquad \mathbf{X} = X_\infty \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}(1 - e^{-\gamma_\alpha \|\boldsymbol{\alpha}\|}) \qquad K = K_\infty(1 - e^{-\gamma_\kappa \kappa}) \tag{19}$$

Note that the laws defined by Eqs. (18) and (19) are subject to the constraint that $\mathbf{X} = \mathbf{0}$ when $\|\boldsymbol{\alpha}\| = 0$. Also note that the internal state variable–thermodynamic conjugate force relationships given by Eqs. (18) and (19) are defined based on the material being investigated, and different relationships can be selected for each of the hardening terms. Though two simple models, the power and exponential laws, are used here, more complex models can be incorporated in the same manner; however, the analysis of various material models is beyond the scope of this work. This work is focused on the development of a formulation based on a general functional form of the thermodynamic conjugate forces. This allows the constitutive model to be developed without making assumptions as to the behavior of the material model, such that the conjugate forces are written as general functions of the internal state variables:

$$R = R(r); \qquad \mathbf{X} = \mathbf{X}(\boldsymbol{\alpha}); \qquad K = K(\kappa) \tag{20}$$

Laplacians of the conjugate forces are required for the gradient-enhanced measures given by Eqs. (3). Defining explicit forms for the energy terms due to the gradient state variables would place constraints on the gradients. Rather than enforce a constraint that may or may not be realistic, expressions for the conjugate force gradients are directly derived from Eqs. (20), as follows:

$$\nabla R = \frac{\partial R(r)}{\partial r}\nabla r; \qquad \nabla^2 R = \frac{\partial^2 R(r)}{\partial r^2}\nabla r \cdot \nabla r + \frac{\partial R(r)}{\partial r}\nabla^2 r \tag{21}$$

$$\nabla \mathbf{X} = \frac{\partial \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} : \nabla \boldsymbol{\alpha}; \qquad \nabla^2 \mathbf{X} = \frac{\partial^2 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}} : \nabla \boldsymbol{\alpha} \cdot \nabla \boldsymbol{\alpha} + \frac{\partial \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} : \nabla^2 \boldsymbol{\alpha} \tag{22}$$

$$\nabla K = \frac{\partial K(\kappa)}{\partial \kappa} : \nabla \kappa; \qquad \nabla^2 K = \frac{\partial^2 K(\kappa)}{\partial \kappa \partial \kappa} : \nabla \kappa \cdot \nabla \kappa + \frac{\partial K(\kappa)}{\partial \kappa} : \nabla^2 \kappa \tag{23}$$

Thus, based on the material model defined by Eqs. (20), the gradient conjugate forces will be readily available without having to introduce additional models for the gradient terms.

E. Dissipation Potential and Flow Rules

The evolution relations of the internal state variables are obtained by assuming the physical existence of a dissipation potential at the macroscale. The theory of functions of several variables is used with a plastic Lagrange multiplier $\dot{\lambda}^p$ and a damage Lagrange multiplier $\dot{\lambda}^d$ to construct the objective function in terms of the plastic potential F and the damage potential G. The objective function is extremized such that, for the case when $F \geq 0$ and $G \geq 0$, the following evolution equations are obtained [14]:

$$\dot{\boldsymbol{\varepsilon}}^{pd} = \frac{\partial F}{\partial \boldsymbol{\sigma}}\dot{\lambda}^p + \frac{\partial G}{\partial \boldsymbol{\sigma}}\dot{\lambda}^d; \qquad \dot{\boldsymbol{\varphi}} = -\frac{\partial F}{\partial \mathbf{Y}}\dot{\lambda}^p - \frac{\partial G}{\partial \mathbf{Y}}\dot{\lambda}^d \tag{24}$$

$$\dot{\mathbf{V}}^p = -\frac{\partial F}{\partial \mathbf{A}^p}\dot{\lambda}^p; \qquad \dot{\mathbf{V}}^d = -\frac{\partial G}{\partial \mathbf{A}^d}\dot{\lambda}^d \tag{25}$$

Because the plastic-damage strain rate will be developed in the current deformed and damaged configuration, its corresponding evolution equation will be a function of the damage measure. Similarly, the evolution equation of the conjugate force due to damage will be a function of the stress. The evolution equations for the inelastic strain and the damage are interdependent [6], and therefore the two dissipative mechanisms given by Eqs. (24) are implicitly interdependent through the stress and the conjugate forces due to damage. Note that if $F \leq 0$, then $\partial F/\partial \boldsymbol{\sigma} = \partial F/\partial \mathbf{Y} = 0$, or if $G \leq 0$, then $\partial G/\partial \boldsymbol{\sigma} = \partial G/\partial \mathbf{Y} = 0$, and the evolution equations for the inelastic strain and the damage become decoupled.

As in the discussion in the previous section on defining the gradient conjugate forces, the evolution equations for the gradients of the internal state variables are defined directly by operating on Eqs. (25), as follows:

$$\nabla \dot{\mathbf{V}}^p = -\nabla\left(\frac{\partial F}{\partial \mathbf{A}^p}\right)\dot{\lambda}^p - \frac{\partial F}{\partial \mathbf{A}^p}\nabla \dot{\lambda}^p \qquad \nabla^2 \dot{\mathbf{V}}^p = -\nabla^2\left(\frac{\partial F}{\partial \mathbf{A}^p}\right)\dot{\lambda}^p - 2\nabla\left(\frac{\partial F}{\partial \mathbf{A}^p}\right)\cdot \nabla \dot{\lambda}^p - \frac{\partial F}{\partial \mathbf{A}^p}\nabla^2 \dot{\lambda}^p \tag{26}$$

$$\nabla \dot{\mathbf{V}}^d = -\nabla\left(\frac{\partial G}{\partial \mathbf{A}^d}\right)\dot{\lambda}^d - \frac{\partial G}{\partial \mathbf{A}^d}\nabla \dot{\lambda}^d \qquad \nabla^2 \dot{\mathbf{V}}^d = -\nabla^2\left(\frac{\partial G}{\partial \mathbf{A}^d}\right)\dot{\lambda}^d - 2\nabla\left(\frac{\partial G}{\partial \mathbf{A}^d}\right)\cdot \nabla \dot{\lambda}^d - \frac{\partial G}{\partial \mathbf{A}^d}\nabla^2 \dot{\lambda}^d \tag{27}$$

To determine the gradients and Laplacians of the normals to the plastic potential, consider a point in the plastic domain for which $F = 0$ at this point. If it is assumed that the surrounding area has also entered into the plastic regime ($F = 0$), then the gradient and Laplacian of the yield condition at this point can be assumed to be zero ($\nabla F = 0$; $\nabla^2 F = 0$). A similar argument can be made for a point in the damage domain, such that the following simplifications are used:

$$\nabla\left(\frac{\partial F}{\partial \mathbf{A}^p}\right) = \frac{\partial(\nabla F)}{\partial \mathbf{A}^p} = 0; \qquad \nabla^2\left(\frac{\partial F}{\partial \mathbf{A}^p}\right) = \frac{\partial(\nabla^2 F)}{\partial \mathbf{A}^p} = 0 \tag{28}$$

$$\nabla\left(\frac{\partial G}{\partial \mathbf{A}^d}\right) = \frac{\partial(\nabla G)}{\partial \mathbf{A}^d} = 0; \qquad \nabla^2\left(\frac{\partial G}{\partial \mathbf{A}^d}\right) = \frac{\partial(\nabla^2 G)}{\partial \mathbf{A}^d} = 0 \tag{29}$$

Thus, the evolution equations for the internal state variables are simplified as follows:

$$\dot{r} = -f_{,R}\dot{\lambda}^p; \qquad \nabla \dot{r} = -f_{,R}\nabla \dot{\lambda}^p; \qquad \nabla^2 \dot{r} = -f_{,R}\nabla^2 \dot{\lambda}^p \tag{30}$$

$$\dot{\boldsymbol{\alpha}} = f_{,\boldsymbol{\sigma}}\dot{\lambda}^p; \qquad \nabla \dot{\boldsymbol{\alpha}} = f_{,\boldsymbol{\sigma}}\nabla \dot{\lambda}^p; \qquad \nabla^2 \dot{\boldsymbol{\alpha}} = f_{,\boldsymbol{\sigma}}\nabla^2 \dot{\lambda}^p \tag{31}$$

$$\dot{\kappa} = -g_{,K}\dot{\lambda}^d; \qquad \nabla \dot{\kappa} = -g_{,K}\nabla \dot{\lambda}^d; \qquad \nabla^2 \dot{\kappa} = -g_{,K}\nabla^2 \dot{\lambda}^d \tag{32}$$

The following loading–unloading conditions, known as the Kuhn–Tucker conditions, must also be enforced:

$$\dot{\lambda}^p \geq 0; \qquad f \leq 0; \qquad \dot{\lambda}^p f = 0 \tag{33}$$

$$\dot{\lambda}^d \geq 0; \qquad g \leq 0; \qquad \dot{\lambda}^d g = 0 \tag{34}$$

F. Yield Condition and Damage Condition

Associative plasticity and associative damage are used here to derive the evolution equations for the constitutive model, such that the plastic potential F is set equal to the yield criterion f, and the damage potential G is set equal to the damage criterion g. Note that the plasticity condition is given in the effective, undamaged configuration. Thus, the gradient-enhanced yield criterion and the gradient-enhanced damage criterion, written in the form of a von Mises criterion, are given as follows:

$$F = f = \|\tilde{\boldsymbol{\xi}}\| - \sqrt{\frac{2}{3}}[\sigma_{yp} + \tilde{\bar{R}}] \leq 0; \qquad G = g = \|\mathbf{Y}\| - \sqrt{\frac{2}{3}}[\sigma_{yd} + \bar{K}] \equiv 0 \tag{35}$$

where σ_{yp} and σ_{yd} are the initial thresholds at which plasticity and damage, respectively, begin to occur. Using the definition of $\tilde{\bar{R}}$ given by Eq. (4) and of $\tilde{\boldsymbol{\xi}}$ given by Eq. (6), the plastic potential can be written in terms of the damaged state, such that:

$$F = f = \|\mathbf{N}:(\boldsymbol{\sigma} - \bar{\mathbf{X}})\| - \sqrt{\frac{2}{3}}\left[\sigma_{yp} + \frac{\bar{R}}{1 - \|\boldsymbol{\varphi}\|}\right] \leq 0 \tag{36}$$

The normals to the plastic potential and to the damage potential can now be defined as given in the Appendix, allowing the evolution equations of the internal state variables to be defined.

G. Plasticity and Damage Consistency Conditions

At a plastic state in which $f = 0$, the consistency condition $\dot{f} = 0$ results from the loading–unloading conditions of Eq. (33). Similarly, at a damage state in which $g = 0$, the consistency condition $\dot{g} = 0$ results from the loading–unloading conditions of Eq. (34). Thus, as the conjugate forces have been defined as functions of the state variables, as shown in Eqs. (21), the consistency conditions can be expanded in terms of the flux variables, as follows:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{\varphi}}:\dot{\boldsymbol{\varphi}} + \frac{\partial f}{\partial R}\frac{\partial R}{\partial r}\dot{r} + \frac{\partial f}{\partial \nabla^2 R}\left(\frac{\partial \nabla^2 R}{\partial r}\dot{r} + \frac{\partial \nabla^2 R}{\partial \nabla r}\cdot\nabla\dot{r} + \frac{\partial \nabla^2 R}{\partial \nabla^2 r}\nabla^2\dot{r}\right) + \frac{\partial f}{\partial \mathbf{X}}:\frac{\partial \mathbf{X}}{\partial \boldsymbol{\alpha}}:\dot{\boldsymbol{\alpha}} + \frac{\partial f}{\partial \nabla^2 \mathbf{X}}:\left(\frac{\partial \nabla^2 \mathbf{X}}{\partial \boldsymbol{\alpha}}:\dot{\boldsymbol{\alpha}} + \frac{\partial \nabla^2 \mathbf{X}}{\partial \nabla \boldsymbol{\alpha}}:\nabla\dot{\boldsymbol{\alpha}} + \frac{\partial \nabla^2 \mathbf{X}}{\partial \nabla^2 \boldsymbol{\alpha}}:\nabla^2\dot{\boldsymbol{\alpha}}\right) \equiv 0 \tag{37}$$

$$\dot{g} = \frac{\partial g}{\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} + \frac{\partial g}{\partial \boldsymbol{\varphi}}:\dot{\boldsymbol{\varphi}} + \frac{\partial g}{\partial K}\frac{\partial K}{\partial \kappa}\dot{\kappa} + \frac{\partial g}{\partial \nabla^2 K}\left(\frac{\partial \nabla^2 K}{\partial \kappa}\dot{\kappa} + \frac{\partial \nabla^2 K}{\partial \nabla \kappa}\cdot\nabla\dot{\kappa} + \frac{\partial \nabla^2 K}{\partial \nabla^2 \kappa}\nabla^2\dot{\kappa}\right) \equiv 0 \tag{38}$$

Note that the derivatives of the Laplacian conjugate forces, with respect to their corresponding state variables and gradient state variables, are also needed and are computed as follows:

$$\frac{\partial \nabla^2 R}{\partial r} = \frac{\partial^3 R(r)}{\partial r^3}\nabla r\cdot\nabla r + \frac{\partial^2 R(r)}{\partial r^2}\nabla^2 r; \qquad \frac{\partial \nabla^2 R}{\partial \nabla r} = 2\frac{\partial^2 R(r)}{\partial r^2}\nabla r; \qquad \frac{\partial \nabla^2 R}{\partial \nabla^2 r} = \frac{\partial R(r)}{\partial r} \tag{39}$$

$$\frac{\partial \nabla^2 \mathbf{X}}{\partial \boldsymbol{\alpha}} = \frac{\partial^3 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}}:\nabla\boldsymbol{\alpha}\cdot\nabla\boldsymbol{\alpha} + \frac{\partial^2 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}}:\nabla^2\boldsymbol{\alpha}; \qquad \frac{\partial \nabla^2 \mathbf{X}}{\partial \nabla \boldsymbol{\alpha}} = 2\frac{\partial^2 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}}:\nabla\boldsymbol{\alpha}; \qquad \frac{\partial \nabla^2 \mathbf{X}}{\partial \nabla^2 \boldsymbol{\alpha}} = \frac{\partial \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \tag{40}$$

$$\frac{\partial \nabla^2 K}{\partial \kappa} = \frac{\partial^3 K(\kappa)}{\partial \kappa^3}\nabla\kappa\cdot\nabla\kappa + \frac{\partial^2 K(\kappa)}{\partial \kappa^2}\nabla^2\kappa; \qquad \frac{\partial \nabla^2 K}{\partial \nabla \kappa} = 2\frac{\partial^2 K(\kappa)}{\partial \kappa^2}\nabla\kappa; \qquad \frac{\partial \nabla^2 K}{\partial \nabla^2 \kappa} = \frac{\partial K(\kappa)}{\partial \kappa} \tag{41}$$

The incremental form of the Hookean stress–strain relation, Eq. (13), is also used in the consistency condition and can be written in the following form:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^e:(\dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^{pd}) + \dot{\mathbf{C}}^e:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{pd}) \tag{42}$$

where the incremental damaged elastic stiffness tensor is found, such that:

$$\dot{\mathbf{C}}^e = \frac{\partial \mathbf{C}^e}{\partial \boldsymbol{\varphi}}:\dot{\boldsymbol{\varphi}} = 2\mathbf{C}^e:\mathbf{M}^T:\frac{\partial \mathbf{M}^{-T}}{\partial \boldsymbol{\varphi}}:\dot{\boldsymbol{\varphi}} \tag{43}$$

Thus, the incremental stress can be written in the following form:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^e:(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{pd}) + \mathbf{C}^e:\mathbf{Z}:\dot{\boldsymbol{\varphi}} \tag{44}$$

where the fourth-order tensor $\mathbf{Z}$ is defined as follows:

$$\mathbf{Z} = 2\mathbf{M}^T:\frac{\partial \mathbf{M}^{-T}}{\partial \boldsymbol{\varphi}}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{pd}) = \mathbf{C}^{-e}:\frac{\partial \mathbf{C}^e}{\partial \boldsymbol{\varphi}}:(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{pd}) \tag{45}$$

After substituting the derivatives [the normals to the yield surface, as defined in the Appendix; the evolution equations for the internal state variables, as defined in Eqs. (30–32); and the incremental Hookean stress–strain relation, Eq. (42)], the consistency conditions are expanded and can be written in the following forms [15,16]:

$$\begin{Bmatrix} h_1^{pp}\dot{\lambda}^p + \mathbf{h}_2^{pp}\cdot\nabla\dot{\lambda}^p + h_3^{pp}\nabla^2\dot{\lambda}^p + h^{pd}\dot{\lambda}^d \\ h_1^{dd}\dot{\lambda}^d + \mathbf{h}_2^{dd}\cdot\nabla\dot{\lambda}^d + h_3^{dd}\nabla^2\dot{\lambda}^d + h^{dp}\dot{\lambda}^p \end{Bmatrix} = \begin{Bmatrix} b^p \\ b^d \end{Bmatrix} \tag{46}$$

where

$$h_1^{pp} = f_{,\boldsymbol{\sigma}}:\mathbf{C}^e:f_{,\boldsymbol{\sigma}} + (f_{,\boldsymbol{\varphi}} + f_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\mathbf{Z}):f_{,\mathbf{Y}} + f_{,R}^2\left(\frac{\partial R(r)}{\partial r} + c_R\frac{\partial^3 R(r)}{\partial r^3}\nabla r\cdot\nabla r + c_R\frac{\partial^2 R(r)}{\partial r^2}\nabla^2 r\right) + f_{,\boldsymbol{\sigma}}:\left(\frac{\partial \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} + c_X\frac{\partial^3 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}}:\nabla\boldsymbol{\alpha}\cdot\nabla\boldsymbol{\alpha} + c_X\frac{\partial^2 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}}:\nabla^2\boldsymbol{\alpha}\right):f_{,\boldsymbol{\sigma}} \tag{47}$$

$$h_1^{dd} = g_{,\boldsymbol{\sigma}}:\mathbf{C}^e:g_{,\boldsymbol{\sigma}} + (g_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\mathbf{Z} + g_{,\boldsymbol{\varphi}}):g_{,\mathbf{Y}} + g_{,K}^2\left(\frac{\partial K(\kappa)}{\partial \kappa} + c_K\frac{\partial^3 K(\kappa)}{\partial \kappa^3}\nabla\kappa\cdot\nabla\kappa + c_K\frac{\partial^2 K(\kappa)}{\partial \kappa^2}\nabla^2\kappa\right) \tag{48}$$

$$\mathbf{h}_2^{pp} = 2c_R f_{,R}^2\frac{\partial^2 R(r)}{\partial r^2}\nabla r + 2c_X f_{,\boldsymbol{\sigma}}:\frac{\partial^2 \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}}:\nabla\boldsymbol{\alpha}:f_{,\boldsymbol{\sigma}}; \qquad \mathbf{h}_2^{dd} = 2c_K g_{,K}^2\frac{\partial^2 K(\kappa)}{\partial \kappa^2}\nabla\kappa \tag{49}$$

$$h_3^{pp} = c_R f_{,R}^2\frac{\partial R(r)}{\partial r} + c_X f_{,\boldsymbol{\sigma}}:\frac{\partial \mathbf{X}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}:f_{,\boldsymbol{\sigma}}; \qquad h_3^{dd} = c_K g_{,K}^2\frac{\partial K(\kappa)}{\partial \kappa} \tag{50}$$

$$h^{pd} = f_{,\boldsymbol{\sigma}}:\mathbf{C}^e:g_{,\boldsymbol{\sigma}} + (f_{,\boldsymbol{\varphi}} + f_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\mathbf{Z}):g_{,\mathbf{Y}}; \qquad h^{dp} = g_{,\boldsymbol{\sigma}}:\mathbf{C}^e:f_{,\boldsymbol{\sigma}} + (g_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\mathbf{Z} + g_{,\boldsymbol{\varphi}}):f_{,\mathbf{Y}} \tag{51}$$

$$b^p = f_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\dot{\boldsymbol{\varepsilon}}; \qquad b^d = g_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\dot{\boldsymbol{\varepsilon}} \tag{52}$$

In a local model using finite elements, the equilibrium equation is required to obtain a unique solution of the boundary-value problem. In this work, the constitutive model contains a set of differential equations that involve macroscale second-order gradients for both the plastic and damage multipliers. To solve such a higher-order problem, additional equations are required. This is done by satisfying both the yield consistency condition and the damage consistency conditions in the weak form. The weak form of the equilibrium equation and of the consistency conditions can be written by using a virtual displacement rate vector $\delta\dot{\mathbf{u}}$, a virtual plastic multiplier rate vector $\delta\dot{\lambda}^p$, and a virtual plastic multiplier rate vector $\delta\dot{\lambda}^d$. In the analysis of a 3-D small deformation plasticity rate boundary-value problem, the body has a plastic region $\mathbf{V}_{\lambda^p}$, and an elastic region outside of this zone. Additionally, the body has a damage region $\mathbf{V}_{\lambda^d}$ which may be in either or both the elastic and plastic domains. Thus, for the governing equations, the equilibrium equation is integrated over the entire body, the plasticity consistency condition is integrated over the plastic region, and the damage consistency condition is integrated over the damage region. Using the consistency conditions in the weak form of the consistency conditions, and integrating (by parts) the weak form of the equilibrium equation, the following governing equations are obtained [15]:

$$\int_{\mathbf{V}} \delta\dot{\boldsymbol{\varepsilon}}:\mathbf{C}^e:[\dot{\boldsymbol{\varepsilon}} - f_{,\boldsymbol{\sigma}}\dot{\lambda}^p - g_{,\boldsymbol{\sigma}}\dot{\lambda}^d - \mathbf{Z}:(f_{,\mathbf{Y}}\dot{\lambda}^p + g_{,\mathbf{Y}}\dot{\lambda}^d)]\,\mathrm{d}\mathbf{V} - \int_{\mathbf{V}} \delta\dot{\mathbf{u}}\cdot\dot{\mathbf{b}}\,\mathrm{d}\mathbf{V} - \int_{\Gamma_t} \delta\dot{\mathbf{u}}\cdot\dot{\hat{\mathbf{t}}}\,\mathrm{d}\Gamma = 0 \tag{53}$$

$$\int_{\mathbf{V}_{\lambda^p}} \delta\dot{\lambda}^p(f_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\dot{\boldsymbol{\varepsilon}} - h_1^{pp}\dot{\lambda}^p - \mathbf{h}_2^{pp}\cdot\nabla\dot{\lambda}^p - h_3^{pp}\nabla^2\dot{\lambda}^p - h^{pd}\dot{\lambda}^d)\,\mathrm{d}\mathbf{V} = 0 \tag{54}$$

$$\int_{\mathbf{V}_{\lambda^d}} \delta\dot{\lambda}^d(g_{,\boldsymbol{\sigma}}:\mathbf{C}^e:\dot{\boldsymbol{\varepsilon}} - h^{dp}\dot{\lambda}^p - h_1^{dd}\dot{\lambda}^d - \mathbf{h}_2^{dd}\cdot\nabla\dot{\lambda}^d - h_3^{dd}\nabla^2\dot{\lambda}^d)\,\mathrm{d}\mathbf{V} = 0 \tag{55}$$

Because the governing equations are now differential equations, an algebraic solution for these multipliers can no longer be obtained, and a finite element solution will be used.

H. Integration Algorithm

In this section, the integration scheme for the gradient-enhanced constitutive model is presented. To solve the boundary-value problem, a multifield approach is adopted such that the plastic and damage multipliers are discretized in addition to the displacement field. In the solution procedure, linearized forms of the governing equations given by Eqs. (53–55) are solved within an incremental iterative Newton–Raphson solution procedure for the increments of strain, plastic multiplier, and damage multipliers over the time increment Δt_j, such that:

$$\boldsymbol{\varepsilon}_j = \boldsymbol{\varepsilon}_0 + \Delta t_j\,\mathrm{d}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \Delta\boldsymbol{\varepsilon}_j; \qquad \Delta\lambda_j^p = \Delta t_j\,\mathrm{d}\lambda^p \qquad \Delta\lambda_j^d = \Delta t_j\,\mathrm{d}\lambda^d \tag{56}$$

where the subscripted *j* and 0 indicate that the variable is computed at iteration *j* and at the previously converged state, respectively; and the symbol Δ denotes a total increment from the previously converged state to the iteration *j*. The increments of the plastic multiplier $\Delta\lambda_j^p$ and of the damage multiplier $\Delta\lambda_j^d$ are then used to update the state of the material. As with the implicit backward Euler scheme presented in [3], the solution of the system of equations is implicit (computed at time *j*) in the plastic strain, the damage measure, the hardening variables, and the plastic flow direction, such that:

$$\boldsymbol{\varepsilon}_j^{pd} = \boldsymbol{\varepsilon}_0^{pd} + \Delta\boldsymbol{\varepsilon}_j^{pd}; \qquad \boldsymbol{\varphi}_j = \boldsymbol{\varphi}_0 + \Delta\boldsymbol{\varphi}_j \tag{57}$$

$$r_j = r_0 - f_{,R}\Delta\lambda_j^p; \qquad \nabla r_j = \nabla r_0 - f_{,R}\Delta\nabla\lambda_j^p \qquad \nabla^2 r_j = \nabla^2 r_0 - f_{,R}\Delta\nabla^2\lambda_j^p \tag{58}$$

$$\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_0 + f_{,\boldsymbol{\sigma}_j}\Delta\lambda_j^p; \qquad \nabla\boldsymbol{\alpha}_j = \nabla\boldsymbol{\alpha}_0 + f_{,\boldsymbol{\sigma}_j}\Delta\nabla\lambda_j^p \qquad \nabla^2\boldsymbol{\alpha}_j = \nabla^2\boldsymbol{\alpha}_0 + f_{,\boldsymbol{\sigma}_j}\Delta\nabla^2\lambda_j^p \tag{59}$$

$$\kappa_j = \kappa_0 - f_{,K}\Delta\lambda_j^d; \qquad \nabla\kappa_j = \nabla\kappa_0 - f_{,K}\Delta\nabla\lambda_j^d \qquad \nabla^2\kappa_j = \nabla^2\kappa_0 - f_{,K}\Delta\nabla^2\lambda_j^d \tag{60}$$

where

$$\Delta\boldsymbol{\varepsilon}_j^{pd} = f_{,\boldsymbol{\sigma}_j}\Delta\lambda_j^p + g_{,\boldsymbol{\sigma}_j}\Delta\lambda_j^d; \qquad \Delta\boldsymbol{\varphi}_j = -f_{,\mathbf{Y}_j}\Delta\lambda_j^p - g_{,\mathbf{Y}_j}\Delta\lambda_j^d \tag{61}$$

The stress and the damage measure for this integration scheme are also defined at iteration *j*, as follows:

$$\boldsymbol{\sigma}_j = \mathbf{C}_j^e:(\boldsymbol{\varepsilon}_j - \boldsymbol{\varepsilon}_j^{pd}) = \boldsymbol{\sigma}_0 + \mathbf{C}_j^e:(\Delta\boldsymbol{\varepsilon}_j - \Delta\boldsymbol{\varepsilon}_j^{pd}) + \mathbf{C}_j^e:\mathbf{Z}_j:\Delta\boldsymbol{\varphi}_j \tag{62}$$

$$\mathbf{Y}_j = \boldsymbol{\sigma}_j^T:\left(\frac{\partial\mathbf{C}^{-e}}{\partial\boldsymbol{\varphi}}\right)_j:\boldsymbol{\sigma}_j \tag{63}$$

The damaged elastic stiffness tensor can also be defined in an incremental form, as follows:

$$\mathbf{C}_j^e = \mathbf{C}_0^e + \Delta\mathbf{C}_j^e \tag{64}$$

where

$$\Delta\mathbf{C}_j^e = \left(\frac{\partial\mathbf{C}^e}{\partial\boldsymbol{\varphi}}\right)_j:\Delta\boldsymbol{\varphi}_j = 2\mathbf{C}_j^e:\mathbf{M}_j^T:\left(\frac{\partial\mathbf{M}^{-T}}{\partial\boldsymbol{\varphi}}\right)_j:\Delta\boldsymbol{\varphi}_j \tag{65}$$

The integration scheme used here enforces that $f_j = 0$ and $g_j = 0$ at the end of the time step. Because the plastic and damage multipliers are available from the nodal degrees of freedom, a return mapping algorithm is not required; however, to compute the stress and the damage conjugate force, an initial elastic-predictor step is used to determine if the point is loaded elastically or plastically and with or without damage. This is then followed by a plastic-damage corrector. In determining the state of the material, the internal state variables are updated using the multipliers in Eqs. (57–60), and the initial trial stress and the initial trial damage conjugate force are computed as follows:

$$\boldsymbol{\sigma}_j^{\mathrm{trial}} = \boldsymbol{\sigma}_0 + \mathbf{C}_0^e:\Delta\boldsymbol{\varepsilon}_j \tag{66}$$

$$\mathbf{Y}_j^{\mathrm{trial}} = \boldsymbol{\sigma}_j^{\mathrm{trial}}:\left(\frac{\partial\mathbf{C}^{-e}}{\partial\boldsymbol{\varphi}}\right)_j:\boldsymbol{\sigma}_j^{\mathrm{trial}} \tag{67}$$

The trial state $(\boldsymbol{\sigma}_j^{\mathrm{trial}}, \mathbf{Y}_j^{\mathrm{trial}}, \boldsymbol{\varepsilon}_j^{pd}, \boldsymbol{\varphi}_j, r_j, \boldsymbol{\alpha}_j, \kappa_j)$ is then used in a trial yield criterion and a trial damage criterion to decide whether an elastic point enters the plastic and/or damage regimes or whether a plastic or damage point elastically unloads. For the case in which $f_{\mathrm{trial}} \leq 0$ and $g_{\mathrm{trial}} \leq 0$, the integration point is assumed to be elastic with no additional damage, and the current state $(\boldsymbol{\sigma}_j, \mathbf{Y}_j, \boldsymbol{\varepsilon}_j^{pd}, \boldsymbol{\varphi}_j, r_j, \boldsymbol{\alpha}_j, \kappa_j)$ is set to the trial state $(\boldsymbol{\sigma}_j^{\mathrm{trial}}, \mathbf{Y}_j^{\mathrm{trial}}, \boldsymbol{\varepsilon}_j^{pd}, \boldsymbol{\varphi}_j, r_j, \boldsymbol{\alpha}_j, \kappa_j)$. Alternatively, when $f_{\mathrm{trial}} > 0$, plasticity has occurred and the stress has to be corrected to the yield surface. Similarly, when $g_{\text{trial}} > 0$, damage has occurred, and the state has to be corrected to the damage surface. Using the definition of the Cauchy stress from Eq. (62) along with the definition of the trial stress, Eq. (66), the Cauchy stress is corrected as follows:

$$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j^{\text{trial}} + \Delta\boldsymbol{\sigma}_j \tag{68}$$

where, after a number of manipulations, the correction to the stress during the corrector phase is defined as follows:

$$\Delta\boldsymbol{\sigma}_j = -\mathbf{C}_j^e{:}\Delta\boldsymbol{\varepsilon}_j^{pd} + \left(\frac{\partial\mathbf{C}^e}{\partial\boldsymbol{\varphi}}\right)_j{:}(\Delta\boldsymbol{\varepsilon}_j + \boldsymbol{\varepsilon}_j - \boldsymbol{\varepsilon}_j^{pd}){:}\Delta\boldsymbol{\varphi}_j \tag{69}$$

Given the increments of strain, plastic multiplier, and damage multiplier from the solution of the global equations, it can be seen that the problem defined by this model can be entirely defined by solving for the increment of the stress $\Delta\boldsymbol{\sigma}_j$.

I. Consistent Tangent Operator

The trial stress can be used to predict if an integration point has entered the plastic and/or damage regime, and the internal state variables can then be updated using the integration scheme. To obtain proper quadratic convergence, the choice of a tangent operator must be consistent with the integration scheme. The consistent tangent operator is defined as follows [2,3]:

$$\mathbf{C}_j^{\text{alg}} = \left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\boldsymbol{\varepsilon}}\right)_j \tag{70}$$

Using the integration scheme of the previous section and after a number of manipulations, the algorithmic relation for the increment of stress can be derived as follows [15]:

$$\begin{aligned}\mathrm{d}\boldsymbol{\sigma}_j = {}& \mathbf{D}^e{:}(\mathrm{d}\boldsymbol{\varepsilon}_j - f_{,\boldsymbol{\sigma}_j}\,\mathrm{d}\lambda_j^p - g_{,\boldsymbol{\sigma}_j}\,\mathrm{d}\lambda_j^d)\\ &+ \mathbf{D}^e{:}\mathbf{Z}_j{:}(-f_{,\mathbf{Y}}d\lambda_j^p - g_{,\mathbf{Y}}\,\mathrm{d}\lambda_j^d)\end{aligned} \tag{71}$$

where the algorithmic stiffness operator is defined as follows:

$$\begin{aligned}\mathbf{D}^{-e} = {}& \mathbf{C}_j^{-e} + f_{,\boldsymbol{\sigma}_j\boldsymbol{\sigma}_j}\Delta\lambda_j^p + g_{,\boldsymbol{\sigma}_j\boldsymbol{\sigma}_j}\Delta\lambda_j^d\\ &+ \mathbf{Z}_j{:}(f_{,\mathbf{Y}_j\boldsymbol{\sigma}_j}\Delta\lambda_j^p + g_{,\mathbf{Y}_j\boldsymbol{\sigma}_j}\Delta\lambda_j^d)\end{aligned} \tag{72}$$

The required normals to the yield surface and the damage surface are defined in the Appendix. To obtain proper convergence, the elastic stiffness tensor $\mathbf{C}^e$ should be replaced by $\mathbf{D}^e$ in the governing equations. In the next section, an algebraic form of the governing equations is formulated using the integration scheme that can be solved within an incremental iterative Newton–Raphson solution.

J. Mixed Finite Element Formulation

To solve a boundary-value problem, a multifield finite element approach is adopted such that the plastic and damage multipliers are discretized in addition to the displacement field. Because the degrees of freedom are increased, two additional governing equations (i.e., the weak forms of the consistency conditions) are required to obtain a solution. Noting that at the end of a converged time step it is enforced that the yield criterion be zero in the plastic domain $\mathbf{V}_{\lambda^p}$ and that the damage criterion be zero in the damage domain $\mathbf{V}_{\lambda^d}$, the governing equations as defined in Eqs. (53–55) for the integration scheme defined in the previous section and for the algorithmic relation, Eq. (71), are written as follows:

$$\begin{aligned}&\int_{\mathbf{V}}\delta\boldsymbol{\varepsilon}{:}\mathbf{D}_j^e{:}[\mathrm{d}\boldsymbol{\varepsilon} - f_{,\boldsymbol{\sigma}_j}\,\mathrm{d}\lambda^p - g_{,\boldsymbol{\sigma}_j}\,\mathrm{d}\lambda^d - \mathbf{Z}_j{:}(f_{,\mathbf{Y}_j}d\lambda^p + g_{,\mathbf{Y}_j}d\lambda^d)]\,\mathrm{d}\mathbf{V}\\ &\quad= \int_{\mathbf{V}}\delta\mathbf{u}{:}\mathbf{b}_j\,\mathrm{d}\mathbf{V} + \int_{\Gamma_t}\delta\mathbf{u}{:}\hat{\mathbf{t}}_j\,\mathrm{d}\Gamma - \int_{\mathbf{V}}\delta\boldsymbol{\varepsilon}{:}\boldsymbol{\sigma}_j\,\mathrm{d}\mathbf{V}\end{aligned} \tag{73}$$

$$\begin{aligned}&\int_{\mathbf{V}_{\lambda^p}}\delta\dot{\lambda}^p(-f_{,\boldsymbol{\sigma}_j}{:}\mathbf{D}_j^e{:}\,\mathrm{d}\boldsymbol{\varepsilon} + h_1^{pp}\,\mathrm{d}\lambda^p + \mathbf{h}_2^{pp}\cdot\mathrm{d}\nabla\lambda^p + h_3^{pp}\,\mathrm{d}\nabla^2\lambda_j^p\\ &\quad+ h^{pd}\,\mathrm{d}\lambda^p)\,\mathrm{d}\mathbf{V} = \int_{\mathbf{V}_{\lambda^p}}\delta\lambda^p f_j\,\mathrm{d}\mathbf{V}\end{aligned} \tag{74}$$

$$\begin{aligned}&\int_{\mathbf{V}_{\lambda^d}}\delta\lambda^d(-g_{,\boldsymbol{\sigma}_j}{:}\mathbf{D}_j^e{:}\,\mathrm{d}\boldsymbol{\varepsilon} + h^{\text{dp}}\,\mathrm{d}\lambda^p + h_1^{dd}\,\mathrm{d}\lambda^d + \mathbf{h}_2^{dd}\cdot\mathrm{d}\nabla\lambda^d\\ &\quad+ h_3^{dd}\,\mathrm{d}\nabla^2\lambda^d)\,\mathrm{d}\mathbf{V} = \int_{\mathbf{V}_{\lambda^d}}\delta\lambda^d g_j\,\mathrm{d}\mathbf{V}\end{aligned} \tag{75}$$

where the coefficients defined by Eqs. (47–51) are used here and are evaluated at time $t = j$.

Note that the governing equations are written over three different domains. The first governing equation is enforced over the entire body, including both the plastic domain and the damage domain, as well as the elastic domain and the undamaged domain; the second governing equation is enforced over only the plastic domain; the third governing equation is enforced over only the damage domain. Note that these three domains can and will overlap, such that a body can have any or all of the following regions: 1) elastic undamaged, 2) elastic damaged, 3) plastic undamaged, and 4) plastic damaged. To enforce the governing equations as given by Eqs. (73–75), different meshes must be used for each equation. Alternatively, if the same mesh is to be used for all of the governing equations (i.e., the equations are to be integrated over the total domain $\mathbf{V}$), then it is enforced that $f_{,\tilde{\sigma}_j} = 0$, $f_{,Y_j} = 0$, and $\mathrm{d}\lambda^p = 0$ in the elastic domain and that $g_{,\tilde{\sigma}_j} = 0$, $g_{,Y_j} = 0$, $g_j = 0$, and $\mathrm{d}\lambda^d = 0$ in the undamaged domain. The governing equations can then be solved across the entire body [13,17].

For the displacement degrees of freedom, the governing equations only involve first-order derivatives of the displacement field (i.e., strains), and so the discretization procedure for the displacement field only requires $C0$ continuous interpolation functions. The displacement is discretized using a set of displacement nodal degrees of freedom contained in the vector $\{a_j\}$. The problem of a gradient-enhanced model, which includes Laplacians, requires higher-order shape functions to allow the Laplacian to be continuous across the element. Based on the selected degrees of freedom for the discretization of the plastic multiplier and of the damage multiplier, a cubic function will be used to interpolate the plastic multiplier and the damage multiplier. Though cubic polynomials can be derived as either C^0 or C^1 elements, C^1 shape functions allow the use of additional boundary conditions in terms of the gradients of the multipliers. The multipliers are discretized using a set of nodal degrees of freedom contained in the vectors $\{\Lambda_j^p\}$ and $\{\Lambda_j^d\}$.

III. Results and Discussion

Ignoring damage, the gradient-enhanced plasticity model was implemented into ABAQUS using the UEL subroutine. The effectiveness of this gradient model is evaluated by studying the mesh-dependence issue in localization problems through numerical examples. The problems considered here will be focused on ill-posed initial boundary-value problems that behave in a strain-softening manner, due to material instabilities and due to structural instabilities.

In each of the problems investigated, the specimen is considered to be on a smooth rigid surface, such that all vertical displacements are zero on the bottom edge. Horizontal displacement is also constrained at the middle node of the bottom boundary to avoid rigid body displacement. The dimensions of the specimen are $B = 60$ mm and $H = 120$ mm. The refined meshes to be analyzed are created by repeatedly bisecting the sides of the quadrilateral elements, thus creating four elements from each original element. The discretizations to be considered for the biaxial compression include meshes of 6×12, 12×24, and 24×48 elements and for the tensile specimen include meshes of 12×24 and 24×48. Solutions will be obtained using elements with eight-node discretization of the displacement field, and integration will be carried out with 2×2 integration points.

Using a linear isotropic hardening material, the material from [17] is used such that the material is assumed to be elastoplastic with linear isotropic softening. The material has an elastic shear modulus μ^e of 4000 $\mathrm{N/mm^2}$ and a Poisson's ratio ν of 0.49, corresponding to a Young's modulus E of $2\mu^e(1+\nu) = 11920\ \mathrm{N/mm^2}$. The linear softening material used in [17] corresponds to the use of a linear isotropic softening law with a softening coefficient H_R of $\frac{3}{2}(-0.1\mu^e) = -600\ \mathrm{N/mm^2}$, a length scale c_R of $(3.0\ \mathrm{mm})^2$, and an initial yield stress σ_{yp} of 100 $\mathrm{N/mm^2}$. For an exponential isotropic hardening material, the material is assumed to be elastoplastic, with an initial yield stress σ_{yp} of 100 $\mathrm{N/mm^2}$, a Poisson's ratio ν of 0.49, and a Young's modulus E of 11920 $\mathrm{N/mm^2}$. The linear isotropic softening model is used with $R_\infty = -100\ \mathrm{N/mm^2}$ and $\gamma_r = 10.0$.

A. Biaxial Compression

A plane strain specimen under a biaxial state of loading is considered here to demonstrate the capability of the gradient model for a 2-D localization problem. This problem has been investigated by numerous authors for the linear softening case of plasticity and is investigated here to verify the model behavior. A 1-mm vertical displacement is applied to the upper edge of the specimen through a rigid, frictionless plate, such that there are no rotations along the upper edge. To create an inhomogeneous loading state such that a shear band is initiated, a 10 by 10 mm area in the bottom left-hand corner of the sample is assigned a yield strength that is reduced by 10%. The imperfect area is the same for each mesh.

1. Linear Isotropic Softening Law

Setting the length scale to $c_R = 9.0\ \mathrm{mm^2}$, the solution of this problem using the gradient-enhanced model demonstrates the effectiveness of the gradients in obtaining mesh-independent results (Fig. 1). The shear band width becomes independent of the mesh used and dependent on the material length scale, as shown in Fig. 2. As the mesh is refined, the width of the shear band remains independent of the mesh size, and the stress-strain behavior becomes independent of the discretization (Fig. 3).

2. Exponential Isotropic Softening Law

Localization for the biaxial compression example is presented here for a material experiencing exponential isotropic plasticity softening, to demonstrate the ability of this work to model nonlinear material behavior. Setting the length scale to $c_R = 9.0\ \mathrm{mm^2}$, the solution of this problem using the gradient-enhanced model demonstrates the effectiveness of the gradients in obtaining mesh-

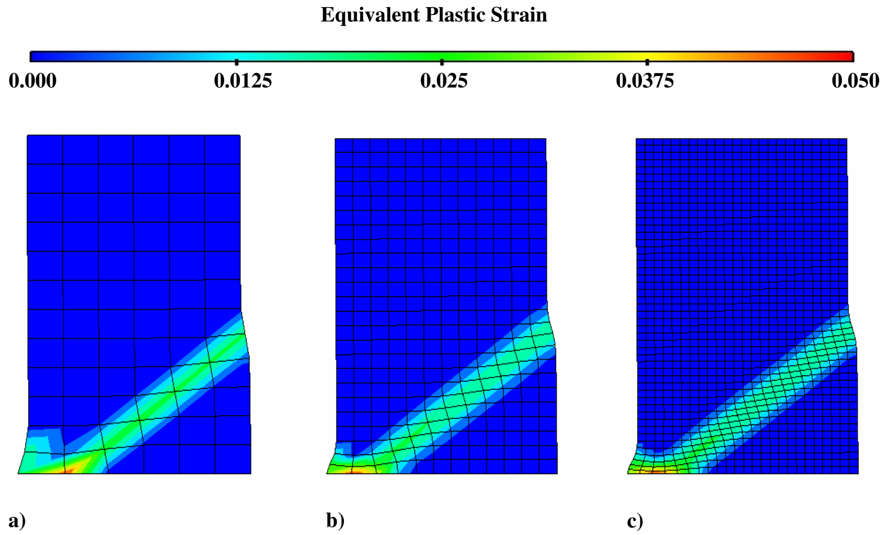

Fig. 1 Mesh independence of the equivalent plastic strain and displacements for the gradient linear softening model. Meshes consist of a) 6 × 12, b) 12 × 24, and c) 24 × 48 elements.

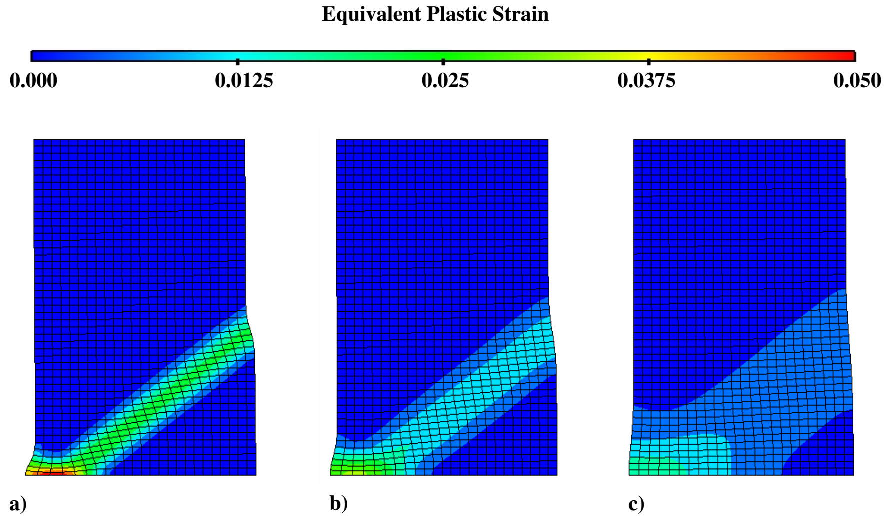

Fig. 2 Length-scale effects on the equivalent plastic strain and displacements for the gradient linear softening model using meshes of 24 × 48 elements; length scales used are a) 9 mm², b) 25 mm², and c) 100 mm².

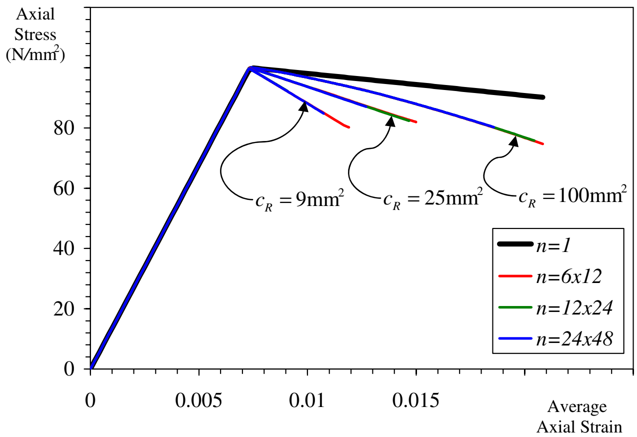

Fig. 3 Solution of length-scale effects on the von Mises stress-average axial strain for the gradient model with linear isotropic softening.

independent results (Fig. 4) for a nonlinear material softening model. The shear band width becomes independent of the mesh used and dependent on the material length scale, as shown in Fig. 5. As the mesh is refined, the width of the shear band remains independent of the mesh size, and the stress-strain behavior becomes independent of the discretization (Fig. 6).

IV. Conclusions

The enhanced continuum model developed in this work provides a strong coupling between gradient-enhanced plasticity and damage and introduces a material length scale into the model. By the introduction of gradients and the corresponding material length scales, the proposed model can properly simulate localization problems without mesh dependency in the numerical solutions. By following a mathematically consistent formulation in the expansion of the Laplacians of the hardening variables, first-order gradients and the Laplacians of several variables are incorporated into the model. The gradient model proposed here consistently expands the Laplacian evolution equations to allow various nonlinear material models.

Typically, researchers make use of the Laplacian to help in the regularization of numerical problems involving patterning and of the first-order gradient for size effects. The proposed model is unique in that 1) a Laplacian-based measure is directly derived as an approximation of a nonlocal measure and 2) the model incorporates first-order up to fourth-order gradients, including odd-ordered gradients. Thus, the proposed capability of this model is increased to account for not just shape patterning, but also to simulate properly size-dependent behavior of the materials.

A computational method for the implementation of this gradient model into a finite element framework has been introduced, and the framework is laid for full implementation for a number of

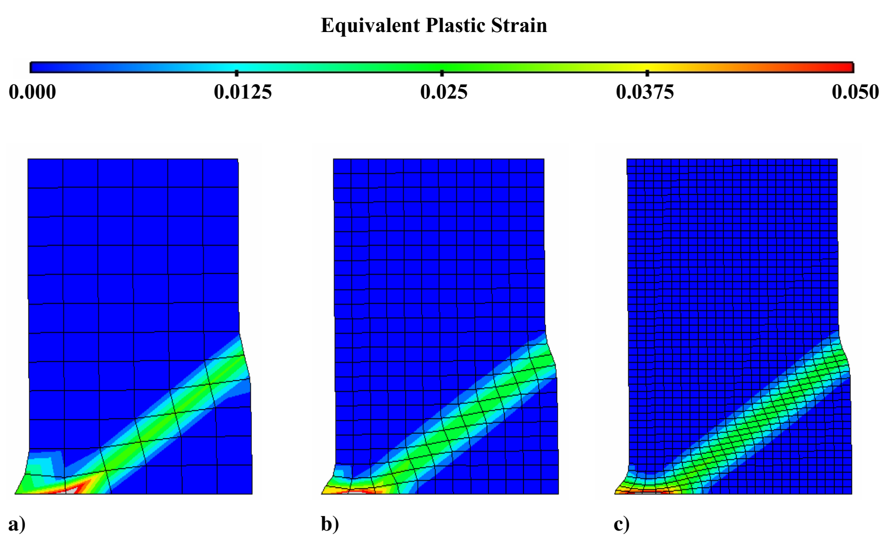

Fig. 4 Mesh independence of the equivalent plastic strain and displacements for the gradient exponential softening model. Meshes consist of a) 6 × 12, b) 12 × 24, and c) 24 × 48 elements.

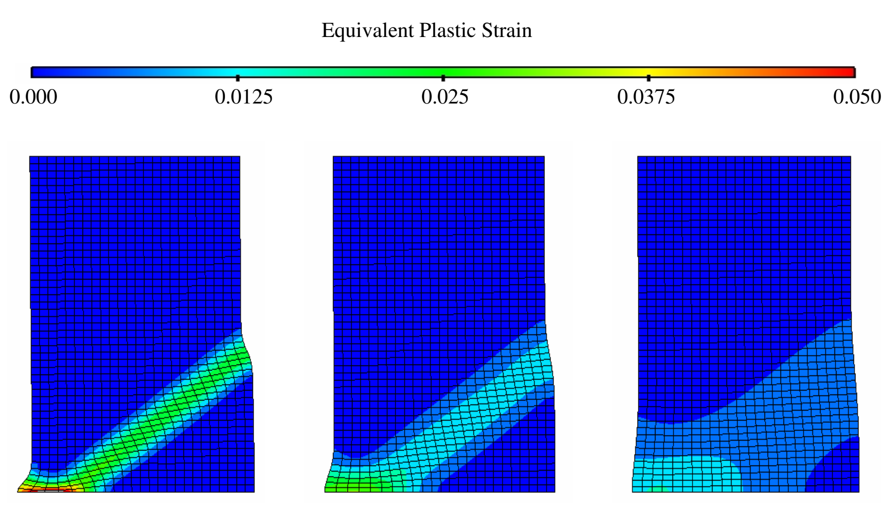

Fig. 5 Length-scale effects on the equivalent plastic strain and displacements for the gradient exponential softening model using meshes of 24 × 48 elements. Length scales used are a) 9 mm^2, b) 25 mm^2, and c) 100 mm^2.

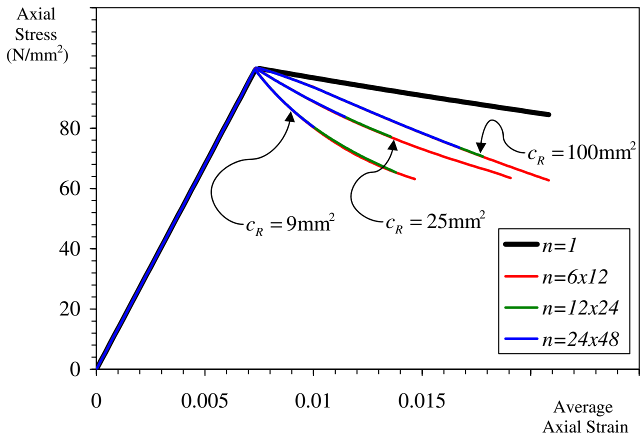

Fig. 6 Solution of length-scale effects on the von Mises stress-average axial strain for the gradient model with exponential isotropic softening.

localization problems. To solve the boundary-value problem, a multifield approach has been adopted such that the plastic and damage multipliers are discretized in addition to the displacement field. Because the degrees of freedom are increased, two additional governing equations (i.e., the weak forms of the consistency conditions) are used to obtain a solution. The discretization procedure for the plastic and damage multipliers uses C^1 continuous interpolation functions, whereas the discretization procedure for the displacement field uses C^0 continuous interpolation functions.

Problems due to a material instability were investigated, and the numerical solutions demonstrated the effectiveness of the gradient theory as a localization limiter and demonstrated how the gradient theory removes the mesh dependency found in classical continuum models.

Appendix: Derivatives of Specific Expressions

To avoid adding additional complexity to the main body of this document, a number of the more complex equations are given here. Indicial notation is used to better demonstrate how the various matrices interact.

Damage-Related Derivatives

$$\frac{\partial\varphi_{ab}}{\partial\hat{\varphi}_{cd}} = Q_{ca}Q_{db} = \frac{\partial\hat{\varphi}_{cd}}{\partial\varphi_{ab}}\frac{\partial M_{abcd}}{\partial\varphi_{wx}} = Q_{ea}Q_{fb}\frac{\partial\hat{M}_{efgh}}{\partial\varphi_{wx}}Q_{gc}Q_{hd}$$

$$\frac{\partial M^{-1}_{abcd}}{\partial\varphi_{ab}} = Q_{ea}Q_{fb}\frac{\partial\hat{M}^{-1}_{efgh}}{\partial\varphi_{ab}}Q_{gc}Q_{hd}$$

$$\frac{\partial^2 M_{abcd}}{\partial\varphi_{wx}\partial\varphi_{yz}} = Q_{ea}Q_{fb}\frac{\partial\hat{M}_{efgh}}{\partial\varphi_{wx}\partial\varphi_{yz}}Q_{gc}Q_{hd}\frac{\partial\hat{M}_{efgh}}{\partial\varphi_{wx}} = \frac{\partial\hat{M}_{efgh}}{\partial\hat{\varphi}_{cd}}\frac{\partial\hat{\varphi}_{cd}}{\partial\varphi_{wx}}$$

$$\frac{\partial^2\hat{M}_{efgh}}{\partial\varphi_{wx}\partial\varphi_{yz}} = \frac{\partial^2\hat{M}_{efgh}}{\partial\hat{\varphi}_{ab}\partial\hat{\varphi}_{cd}}\frac{\partial\hat{\varphi}_{ab}}{\partial\varphi_{wx}}\frac{\partial\hat{\varphi}_{cd}}{\partial\varphi_{yz}}\frac{\partial\hat{M}^{-1}_{efgh}}{\partial\varphi_{wx}} = \frac{\partial\hat{M}^{-1}_{efgh}}{\partial\hat{\varphi}_{cd}}\frac{\partial\hat{\varphi}_{cd}}{\partial\varphi_{wx}}$$

$$\frac{\partial^2\hat{M}^{-1}_{efgh}}{\partial\varphi_{wx}\partial\varphi_{yz}} = 0\,\frac{\partial N_{abcd}}{\partial M_{wxyz}} = \left(\delta_{aw}\delta_{bx} - \frac{1}{3}\delta_{ab}\delta_{wx}\right)\delta_{cy}\delta_{dz}$$

$$\frac{\partial N_{abcd}}{\partial Y_{yz}} = \frac{\partial N_{abcd}}{\partial\varphi_{ef}}\left(\frac{\partial Y_{yz}}{\partial\varphi_{ef}}\right)^{-1}\frac{\partial N_{abcd}}{\partial\varphi_{yz}}$$

$$= \left(\delta_{ae}\delta_{bf} - \frac{1}{3}\delta_{ab}\delta_{ef}\right)\frac{\partial M_{efcd}}{\partial\varphi_{yz}}$$

$$\frac{\partial^2 N_{abcd}}{\partial\varphi_{wx}\partial\varphi_{yz}} = \left(\delta_{ae}\delta_{bf} - \frac{1}{3}\delta_{ab}\delta_{ef}\right)\frac{\partial^2 M_{efcd}}{\partial\varphi_{wx}\partial\varphi_{yz}}\frac{\partial Y_{ab}}{\partial\sigma_{yz}} = \sigma_{cd}\frac{\partial C^{-e}_{cdyz}}{\partial\varphi_{ab}}$$

$$\frac{\partial^2 Y_{ab}}{\partial\sigma_{wx}\partial\sigma_{yz}} = \frac{\partial C^{-e}_{wxyz}}{\partial\varphi_{ab}}$$

$$\frac{\partial Y_{ab}}{\partial\varphi_{yz}} = \left(\frac{\partial\varphi_{ab}}{\partial Y_{yz}}\right)^{-1} = \frac{1}{2}\varepsilon^e_{cd}\frac{\partial^2 C^e_{cdef}}{\partial\varphi_{ab}\partial\varphi_{yz}}\varepsilon^e_{ef}$$

Elasticity Modulus and Related Derivatives

$$\frac{\partial C^e_{abcd}}{\partial\varphi_{yz}} = 2M^{-1}_{abef}\tilde{C}^e_{efgh}\frac{\partial M^{-1}_{cdgh}}{\partial\varphi_{yz}} = 2C^e_{abcd}M_{ghcd}\frac{\partial M^{-1}_{cdgh}}{\partial\varphi_{yz}}$$

$$\frac{\partial^2 C^e_{abcd}}{\partial\varphi_{wx}\partial\varphi_{yz}} = 2\frac{\partial M^{-1}_{abef}}{\partial\varphi_{wx}}\tilde{C}^e_{efgh}\frac{\partial M^{-1}_{cdgh}}{\partial\varphi_{yz}}\frac{\partial C^{-e}_{abcd}}{\partial\varphi_{yz}}$$

$$= 2M_{efab}\tilde{C}^{-e}_{efgh}\frac{\partial M_{ghcd}}{\partial\varphi_{yz}}$$

$$\frac{\partial^2 C^{-e}_{abcd}}{\partial\varphi_{wx}\partial\varphi_{yz}} = 2\frac{\partial M_{efab}}{\partial\varphi_{wx}}\tilde{C}^{-e}_{efgh}\frac{\partial M_{ghcd}}{\partial\varphi_{yz}} + 2M_{efab}\tilde{C}^{-e}_{efgh}\frac{\partial^2 M_{ghcd}}{\partial\varphi_{wx}\partial\varphi_{yz}}$$

Yield Condition and Damage Condition Derivatives

$$\frac{\partial F}{\partial\sigma_{yz}} = f_{,\sigma_{yz}} = \frac{N_{abyz}N_{abef}(\sigma_{ef} - \bar{X}_{ef})}{\|\mathbf{N}{:}(\boldsymbol{\sigma} - \bar{\mathbf{X}})\|}\frac{\partial F}{\partial R} = f_{,R} = \frac{-\sqrt{\frac{2}{3}}}{1 - \|\boldsymbol{\varphi}\|}$$

$$\frac{\partial F}{\partial\nabla^2 R} = f_{,\nabla^2 R} = c_R f_{,R}$$

$$\frac{\partial F}{\partial\nabla^2 X_{yz}} = f_{,\nabla^2 X_{yz}} = -c_X f_{,\sigma_{yz}}\frac{\partial F}{\partial\varphi_{ab}} = f_{,\varphi_{ab}}$$

$$= \frac{\partial N_{ghcd}}{\partial\varphi_{ab}}\frac{(\sigma_{cd} - \bar{X}_{cd})N_{ghef}(\sigma_{ef} - \bar{X}_{ef})}{\|\mathbf{N}{:}(\boldsymbol{\sigma} - \bar{\mathbf{X}})\|}$$

$$- \sqrt{\frac{2}{3}}\frac{\bar{R}\varphi_{ab}}{\|\boldsymbol{\varphi}\|(1 - \|\boldsymbol{\varphi}\|)^2}$$

$$\frac{\partial F}{\partial Y_{yz}} = f_{,Y_{yz}} = f_{,\varphi_{ab}}\left(\frac{\partial Y_{yz}}{\partial\varphi_{ab}}\right)^{-1}\frac{\partial G}{\partial\sigma_{yz}} = g_{,\sigma_{yz}} = g_{,Y_{ab}}\frac{\partial Y_{ab}}{\partial\sigma_{yz}}\frac{\partial G}{\partial K}$$

$$= g_{,K} = -\sqrt{\frac{2}{3}}$$

$$\frac{\partial G}{\partial\nabla^2 K} = g_{,\nabla^2 K} = c_K g_{,K}\frac{\partial G}{\partial\varphi_{yz}} = g_{,\varphi_{yz}} = g_{,Y_{ab}}\frac{\partial Y_{ab}}{\partial\varphi_{yz}}$$

$$\frac{\partial G}{\partial Y_{yz}} = g_{,Y_{yz}} = \frac{Y_{yz}}{\|\mathbf{Y}\|}$$

Yield Condition and Damage Condition Second Derivatives

$$\frac{\partial^2 F}{\partial\sigma_{wx}\partial\sigma_{yz}} = f_{,\sigma_{wx}\sigma_{yz}} = \frac{N_{abwx}N_{abyz} - f_{,\sigma_{wx}}f_{,\sigma_{yz}}}{\|\mathbf{N}{:}(\boldsymbol{\sigma} - \bar{\mathbf{X}})\|}$$

$$\frac{\partial^2 F}{\partial Y_{wx}\partial\sigma_{yz}} = f_{,Y_{wx}\sigma_{yz}} = f_{,\sigma_{yz}N_{abcd}}\frac{\partial N_{abcd}}{\partial Y_{wx}}\frac{\partial^2 F}{\partial\sigma_{yz}\partial N_{klmn}} = f_{,\sigma_{yz}N_{klmn}}$$

$$= \frac{N_{klyz}(\sigma_{mn} - \bar{X}_{mn}) + \delta_{ym}\delta_{zn}N_{klij}(\sigma_{ij} - \bar{X}_{ij})}{\|\mathbf{N}{:}(\boldsymbol{\sigma} - \bar{\mathbf{X}})\|}$$

$$- \frac{f_{,\sigma_{yz}}[(\sigma_{mn} - \bar{X}_{mn})N_{klcd}(\sigma_{cd} - \bar{X}_{cd})]}{\|\mathbf{N}{:}(\boldsymbol{\sigma} - \bar{\mathbf{X}})\|^2}\frac{\partial^2 G}{\partial\sigma_{wx}\partial\sigma_{yz}} = g_{,\sigma_{wx}\sigma_{yz}}$$

$$= g_{,Y_{ab}\sigma_{wx}}\frac{\partial Y_{ab}}{\partial\sigma_{yz}} + g_{,Y_{ab}}\frac{\partial^2 Y_{ab}}{\partial\sigma_{wx}\partial\sigma_{yz}}$$

$$\frac{\partial^2 G}{\partial Y_{wx}\partial\sigma_{yz}} = g_{,Y_{wx}\sigma_{yz}} = g_{,Y_{wx}Y_{ab}}\frac{\partial Y_{ab}}{\partial\sigma_{yz}}$$

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